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COMMENTS

COMMENT ON 'GENERALIZED POTENTIAL FLOW THEORY AND DIRECT CALCULATION OF VELOCITIES APPLIED TO THE NUMERICAL SOLUTION OF THE NAVIER–STOKES AND THE BOUSSINESQ EQUATIONS'

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SUMMARY

In a recent paper a generalized potential flow theory and its application to the solution of the Navier–Stokes equation are developed.[1] The purpose of this comment is to show that the analysis presented in that paper is in general not correct. We note that the theoretical development of Reference 1 is in fact an extension—although not cited—of some work first done by Hawthorne for steady inviscid flow.[2] Hawthorne's solution is correct, and his analysis, which we briefly describe, provides a useful introduction to this note.

KEY WORDS Navier–Stokes equation

HAWTHORNE'S SOLUTION

The momentum equation for steady, inviscid, incompressible flow is

$$\mathbf{v} \times \boldsymbol{\omega} = \nabla P_0/\rho, \tag{1}$$

where $\mathbf{v}$ is the velocity vector, $\boldsymbol{\omega}$ is the vorticity vector, ρ is the density and P_0 is the total pressure.

By assuming

$$\boldsymbol{\omega} = \nabla m \times \nabla \psi, \tag{2}$$

the momentum equation (1) is written as

$$\mathbf{v} \times (\nabla m \times \nabla \psi) = \nabla P_0/\rho$$

or

$$(\mathbf{v} \cdot \nabla \psi)\nabla m - (\mathbf{v} \cdot \nabla m)\nabla \psi = \nabla P_0/\rho. \tag{3}$$

A solution for equation (3) is

$$\psi = P_0/\rho. \tag{4}$$

By substituting equation (4) into equation (3), one obtains

$$(\mathbf{v} \cdot \nabla P_0/\rho)\nabla m - (\mathbf{v} \cdot \nabla m)\nabla \frac{P_0}{\rho} = \nabla \frac{P_0}{\rho}. \tag{5}$$

Since

$$\mathbf{v} \cdot \nabla P_0/\rho = 0 \quad \text{(Bernoulli's equation)}, \tag{5a}$$

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then

$$\mathbf{v}\cdot\nabla m=-1 \tag{5b}$$

where m is the drift function. A line of constant m is a vortex filament since

$$\boldsymbol{\omega}\cdot\nabla m=0.$$

It follows from equation (2) that

$$\mathbf{v}=m\nabla P_0/\rho+\nabla\phi \tag{6}$$

By substituting equation (6) in the continuity equation

$$\nabla\cdot\mathbf{v}=0, \tag{7}$$

an equation for ϕ is obtained:

$$\nabla^2\phi=-\nabla\cdot(m\nabla P_0/\rho) \tag{8}$$

Equations (5a), (5b) and (8) with proper boundary conditions determine the flow field completely.

NAVIER–STOKES EQUATION

The incompressible Navier–Stokes equation is written in terms of the vorticity $\boldsymbol{\omega}$ as follows:

$$\mathbf{v}\times\boldsymbol{\omega}=\nabla P_0/\rho-\nu\nabla\times\boldsymbol{\omega}, \tag{9}$$

where ν is the fluid kinematic viscosity.

Application of Hawthorne's approach to equation (9) is possible if the viscous term can be written as

$$\nabla\times\boldsymbol{\omega}=\mathbf{F}\times\boldsymbol{\omega}, \tag{10}$$

where $\mathbf{F}$ is a vector to be determined later.

Substituting equation (10) into equation (9), we obtain

$$(\mathbf{v}+\nu\mathbf{F})\times\boldsymbol{\omega}=\nabla P_0/\rho.$$

Using equation (2), the above equation is

$$(\mathbf{v}+\nu\mathbf{F})\times(\nabla m\times\nabla\psi)=\nabla P_0/\rho$$

or

$$[(\mathbf{v}+\nu\mathbf{F})\cdot\nabla\psi]\nabla m-[(\mathbf{v}+\nu\mathbf{F})\cdot\nabla m]\nabla\psi=\nabla P_0/\rho. \tag{11}$$

Assuming

$$\psi=P_0/\rho, \tag{12}$$

then

$$(\mathbf{v}+\nu\mathbf{F})\cdot\nabla m=-1 \tag{13a}$$

and

$$(\mathbf{v}+\nu\mathbf{F})\cdot\nabla\psi=0. \tag{13b}$$

Zijl[1] derived the following expressions for $\mathbf{F}$:

$$\mathbf{F}\cdot\nabla m=-(\nabla^2 m+b), \tag{14a}$$

$$\mathbf{F}\cdot\nabla\psi=-(\nabla^2\psi+\beta), \tag{14b}$$

where

$$b=-\frac{[((\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m)\times\nabla m]\cdot(\nabla m\times\nabla\psi)}{(\nabla m\times\nabla\psi)\cdot(\nabla m\times\nabla\psi)} \tag{15a}$$

and

$$\beta=\frac{[((\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m)\times\nabla\psi]\cdot(\nabla m\times\nabla\psi)}{(\nabla m\times\nabla\psi)\cdot(\nabla m\times\nabla\psi)}. \tag{15b}$$

Equation (12) corresponds to equation (11) in Reference 1 and equations (13) and (14) correspond to equations (10a) and (10b) in Reference 1.

We conclude from the above discussion that Zijl's theory[1] is an extension of Hawthorne's solution, for steady inviscid flow, to the Navier–Stokes equation.

Next we investigate the validity of equation (10). If indeed there exists a vector **F** such that

$$\mathbf{F}\times\boldsymbol{\omega}=\nabla\times\boldsymbol{\omega}, \tag{16}$$

this implies that

$$\boldsymbol{\omega}\cdot\nabla\times\boldsymbol{\omega}=0. \tag{17}$$

Equation (17) is in fact not true for three-dimensional flow but only for two-dimensional flow. Thus the above analysis cannot be used for the solution of the Navier–Stokes equation in three dimensions.

Although Zijl[1] followed a slightly different route in his analysis, he arrived at the same result. Zijl derived the following expressions for curl $\boldsymbol{\omega}$:

$$\nabla\times\boldsymbol{\omega}=(\nabla^2\psi)\nabla m-(\nabla^2 m)\nabla\psi+(\nabla\psi\cdot\nabla)\nabla m-(\nabla m\cdot\nabla)\nabla\psi \tag{18}$$

and

$$\nabla\times\boldsymbol{\omega}=(\nabla^2\psi+\beta)\nabla m-(\nabla^2 m+b)\nabla\psi. \tag{19}$$

It is obvious from equation (18) that

$$\boldsymbol{\omega}\cdot\nabla\times\boldsymbol{\omega}\neq 0$$

and from equation (19) that

$$\boldsymbol{\omega}\cdot\nabla\times\boldsymbol{\omega}=0.$$

Thus we conclude that equation (19) is not correct. Consequently, the generalized potential flow theory is not applicable to the solution of the Navier–Stokes equation in three dimensions.

The numerical results obtained for the two-dimensional driven cavity problem in Reference 1 are not computed using Zijl's equations (10a) and (10b).

In addition, the curl of equation (33) in Reference 1 is not equation (34). The correct answer is

$$\nabla(\nu\nabla^2 m)\times\nabla Z=0. \tag{20}$$

REFERENCES

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